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Power law distributions and dynamic behaviour of stock markets

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Abstract. A simple agent model is introduced by analogy with the mean field approach to the Ising model for a magnetic system. Our model is characterised by a generalised Langevin equation $\dot{\varphi} = F(\varphi) + G(\varphi)\,\hat{\eta}(t)$ where $\hat{\eta}(t)$ is the usual Gaussian white noise, *i.e.*: $\langle \hat{\eta}(t)\,\hat{\eta}(t') \rangle = 2D\delta(t - t')$ and $\langle \hat{\eta}(t) \rangle = 0$. Both the associated Fokker Planck equation and the long time probability distribution function can be obtained analytically. A steady state solution may be expressed as $P(\varphi) = \frac{1}{Z}\exp\{-\Psi(\varphi) - \ln G(\varphi)\}$ where $\Psi(\varphi) = -\frac{1}{D}\int^{\varphi} F/(G)^2\, \mathrm{d}\varphi$ and Z is a normalization factor. This is explored for the simple case where $F(\varphi) = J\varphi + b\varphi^2 - c\varphi^3$ and fluctuations characterised by the amplitude $G(\varphi) = \varphi + \varepsilon$ when it readily yields for $\varphi \gg \varepsilon$, a distribution function with power law tails, *viz*: $P(\varphi) = \frac{1}{Z|\varphi|^{1-\frac{J}{D}}}\exp\{(2b\varphi - c\varphi^2)/D\}$. The parameter c ensures convergence of the distribution function for large values of φ. It might be loosely associated with the activity of so-called value traders. The parameter J may be associated with the activity of noise traders. Output for the associated time series show all the characteristics of familiar financial time series providing $J < 0$ and $D \approx |J|$.

PACS. 05.10.Gg Stochastic analysis methods (Fokker-Planck, Langevin, etc.) – 89.65.Gh Economics, business, and financial markets

1 Introduction

Understanding the distribution of financial price fluctuations is an active topic for physicists [1,2]. It is usual to introduce the log-return $s_\tau(t) = \ln p(t + \tau) - \ln p(t)$ where $p(t)$ is the price at time t. If price changes are independent and identically distributed with a well-defined second moment, it follows from the central limit theorem that the distribution function, $P(s)$ converges to a normal distribution for large values of τ. However, for smaller values of τ, there are strong deviations from normal behavior. On close examination, the distribution function, $P(s)$, for the log-returns has so-called fat tails. Mandelbrot [3] suggested that the distribution had characteristics of a Levy distribution, $L_\mu(s)$. For large values of the argument, Levy distributions show the characteristic Pareto tail that for a symmetric distribution may be written as follows:

$$P(s) = L_\mu(s) = A(\mu)/|s|^{1+\mu} \tag{1}$$

The parameter μ is limited to the range $0 < \mu \leqslant 2$. If $\mu = 2$ then the Levy distribution reduces to a normal distribution. Pareto tails may, of course, be defined for values of $\mu > 2$. However such functions are not Levy distributions. For $\mu > 2$ the Levy distribution can take on negative values and is not a stable probability function. Mandelbrot [3] and Fama [4] concluded from studies of daily fluctuations of commodity prices that the associated distribution function was a stable Levy distribution with $\mu \sim 1.7$. Farmer [5] has recently reviewed the position. With the advent of very large data sets taken at ever decreasing time intervals, a number of investigations are showing that as τ increases, the distribution becomes progressively closer to a Gaussian or normal distribution. For large returns, however, the distribution does indeed follow a power law, $P(s) \approx 1/|s|^{1+\mu}$ with $\mu > 2$. The distribution, thus, seems not to be compatible with a Levy distribution. Some results have recently been published by Stanley and his colleagues [6,7]. Approximately 40 million records were included in their data set. As a result they were able to examine the power law scaling over approximately 90 standard deviations and concluded that the cumulative distribution function satisfies a power law with $\mu \sim 3$, implying that the probability distribution satisfies a power law ~ 4. Power law distributions introduced by Pareto in the 19th century are clearly approximations. The distribution function is not integrable at $s = 0$ and is not normalizable if the exponent $\mu < 1$. A power law cannot therefore define the exact distribution function for a variable with an unbounded range. It can be an approximation valid for certain limits.

A number of authors [8–10] have developed dynamic models of the trading process where the market is assumed to arise from the cumulative actions of individual traders. Numerical simulation has then been used to show that such models exhibit distribution functions with

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Pareto-like tails. Such numerical solutions are very powerful and are revealing other aspects of the stochastic processes involved in trading dynamics. However models capable of analytic solution, if these can be found, generally offer greater insight and understanding. Sorin has recently shown that a model where agents interact *via* the classic Lotka-Volterra equations can be explored analytically and demonstrated clearly how power law distribution functions follow from this model. Here we present an approach based on a generalized Langevin equation that yields such an analytic form for the distribution function. The solution shows characteristics typical of stock price distributions including a power law tail regime.

2 Phenomenology for financial markets

The instantaneous price return, $s(t)$, may be expressed as a function of demand, φ. Thus $s(t) = \Im(\varphi)$ where $\Im$ is an increasing function and $\Im(0) = 0$. We follow Bouchaud and Cont [8] and assume that this function can be linearised such that $s(t) = \varphi(t)/\lambda$ where λ is a measure of market depth or liquidity. We assume, with others, that agents respond to a force that has its origin in the action of other traders, *i.e.* $\dot{\varphi}_i(t) = \sum_j f_{ij}(t)$. In general, agents do not have detailed data relating to the specific action of other individual agents and it seems plausible to assume that agents respond to aggregate demand. Thus we might propose that: $f_{ij} = f_i$. We make a further simplification and introduce, by analogy with molecular systems a force of mean demand, $F(\varphi|t) = \sum f_i(\varphi)$ and simply write

$$\dot{\varphi}(t) = F(\varphi|t) \tag{2}$$

The overall instantaneous time dependent demand function, F, thus takes into account, the actions of all agents, be they noise traders who follow the herd, contrary traders, fundamental or value traders, option traders, etc. The time dependence may change slowly as market sentiment changes. Equally it may change in a more random manner and it is this random behaviour that concerns us here. It seems not unreasonable to suppose that the force responsible for demand will have a stochastic element. Anyone who has watched share prices, especially during periods of high volatility will recognize the way their approach to buy and sell decisions can be stressed. In addition, the total force of demand will depend on a trading volume that is of a stochastic character. Assuming the stochastic process is Markovian, we account for this *via* a generalised Langevin equation:

$$\dot{\varphi} = F(\varphi) + G(\varphi)\eta(t) \tag{3}$$

where the random variable η is a Gaussian white noise:

$$\langle \eta(t)\,\eta(t+t')\rangle = 2D\delta(t-t') \quad \text{and } \langle \eta(t)\rangle = 0. \tag{4}$$

We show in the appendix that the distribution function, $P(\varphi,t) = \langle \rho(\varphi,t|\eta\rangle_\eta$ satisfies the generalized Fokker-Planck equation

$$\frac{\partial P}{\partial t} = D\frac{\partial}{\partial \varphi}(G\frac{\partial}{\partial \varphi}(GP)) - \frac{\partial}{\partial \varphi}(FP). \tag{5}$$

This has the steady state solution

$$P(\varphi) = \frac{1}{Z|G(\varphi)|}\exp[-\Psi(\varphi)] \tag{6}$$

$$\Psi(\varphi) = -\frac{1}{D}\int^{\varphi} \frac{F}{G^2}\mathrm{d}\varphi \tag{7}$$

and Z is a normalisation factor.

3 Illustrative example

The simplest approximation to choose is $G(\varphi) = \varphi$. Solomon [10] has shown that if $F(\varphi) = 0$ then $P(\varphi) \sim 1/\varphi$. This is not a good distribution function in the sense that it is not normalisable and we return to this point below. $F(\varphi)$ will in general not be zero. Our initial supposition that there is interaction between agents will also give rise to terms proportional to φ.

Assume now that

$$F(\varphi) = J\varphi + b\varphi^2 - c\varphi^3. \tag{8}$$

The first term on the RHS of equation is associated with direct agent interactions. The second term gives rise to terms of the kind proposed by Solomon who used the Lotka-Volterra approach to population dynamics. If φ were always positive such a term would be sufficient to obtain a distribution function that is well behaved for large values of φ. However we admit negative values for φ since the demand can be both positive and negative. To ensure that the distribution function is well behaved for large negative values of φ we include terms of $O(\varphi^3)$. This has the virtue that our toy financial system cannot go bankrupt.We ignore here the discussion of other more realistic and potentially complex options. If the force F is derived from a potential V this is equivalent to including terms of $O(\varphi^4)$ in the potential – a procedure familiar to physicists versed in theories of critical phenomena. (Indeed in the case that F can be derived from a potential function, V, the Langevin methodology can be constructed within the framework of a Hermitian Hamiltonian, albeit one that is dissipative) [12]. In any event, we obtain:

$$P(\varphi) = \frac{1}{Z|\varphi|}\exp[\frac{1}{D}\int^{\varphi}\frac{(J\varphi + b\varphi^2 - c\varphi^3)}{\varphi^2}] \tag{9}$$

$$= \frac{1}{Z|\varphi|^{1-\frac{J}{D}}}\exp\{(2b\varphi - c\varphi 2)/D\}. \tag{10}$$

When $1 - J/D < 1$ this function clearly cannot be normalized. The arises from using the approximation $G(\varphi)= \varphi$. and can be resolved by choosing $G(\varphi)= \varphi + \varepsilon$ where ε is a small parameter that allows the model to take up the familiar Langevin form as $\varphi \to 0$. The essential outcome is that $P(\varphi)$ will reach a maximum value and no longer diverge as $\varphi \to 0$. Clearly when the exponent of the exponential is small, power laws emerge naturally from this approach. The extent of the deviation of the power law from unity depends on the ratio of the coupling parameter, J to its uncertainty as specified by the parameter D. Furthermore the exponent can take a range of values depending on the values of J and D. In order that the power law is greater than 1, as was found by Mandelbrot and Stanley, it is necessary that $J < 0$. However there is no a priori reason why J should not be positive although such a distribution function would describe a system with an unusual kind of mean value.

4 Discussion

A number of authors have now published numerical simulations that illustrate power law tails [13,14] Our simple model seems to reproduce features found in real markets. The power law tails appear naturally as a result of our introduction of *amplified* fluctuations within the Langevin approach. Numerical results for the time series, that we shall publish elsewhere, appear to add further support to the model. Elsewhere we shall also examine the time correlation functions for $\varphi(t)$ and the volatility $\varphi^2(t)$. The model is also an Ito stochastic process and it is also possible to apply Ito's lemma and explore the consequences for option pricing and volatility smiles. Of particular interest to a physicist is a greater understanding of the fundamental origin of stochastic models of the kind explored here and their position within the general framework of statistical mechanics. This we shall also explore elsewhere.

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Appendix

If

$$\frac{\mathrm{d}\dot{z}}{\mathrm{d}t} = F(z) + G(z)\eta(t). \tag{A.1}$$

The distribution function, $\rho(z,t)$ satisfies

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\dot{z}\rho)}{\partial z} = 0 \tag{A.2}$$

This may be written

$$\frac{\partial \rho}{\partial t} = -L_0\rho - L_1\rho \tag{A.3}$$

where

$$L_0 = \frac{\partial F}{\partial z} + F\frac{\partial}{\partial z} \text{ and } L_1 = \eta\left(\frac{\partial G}{\partial z} + G\frac{\partial}{\partial z}\right). \tag{A.4}$$

Now introduce $\sigma(t)$ such that $\rho(t) = \mathrm{e}^{-L_0 t}\sigma(t)$. It follows that

$$\frac{\partial \sigma}{\partial t} = -V(t)\sigma \text{ where } V(t) = \mathrm{e}^{L_0 t}L_1\mathrm{e}^{-L_0 t} \tag{A.5}$$

This equation has the formal solution

$$\sigma(t) = \exp\left[-\int_0^t \mathrm{d}t' V(t)\right]\sigma(0). \tag{A.6}$$

We expand the exponential to obtain

$$\sigma(t) = \sum_{n=0}^{\infty}\frac{1}{n!}\int_0^t ... \int \mathrm{d}t_1 ... \mathrm{d}t_n V(t_1)...V(t_n)\sigma(0). \tag{A.7}$$

Now take the average over η and use

$$\langle \eta(t)\,\eta(t+t')\rangle = 2D\delta(t') \quad \text{and } \langle \eta(t)\rangle = 0 \tag{A.8}$$

All odd terms are zero and even terms decompose into $2n!/n!2^n$ identical terms each containing a product of pairwise averages $\left\langle \int\int \mathrm{d}t\mathrm{d}t' V(t)V(t')\right\rangle$. As a result we obtain

$$\langle \sigma(t)\rangle_\eta = \sum \frac{(-1)^{2n}}{2n!}\left(\frac{2n!}{n!2^n}\right) \times \int\int \mathrm{d}t\mathrm{d}t' \langle V_t V_{t'}\rangle_\eta \sigma(0) \tag{A.9}$$

$$= \exp\left[\frac{1}{2}\int\int \mathrm{d}t\mathrm{d}t' \langle V_t V_{t'}\rangle_\eta\right]\sigma(0). \tag{A.10}$$

The exponent can now be computed and shown to reduce to

$$D\int_0^t \mathrm{d}t' \mathrm{e}^{L_0 t'}\left\{\frac{\partial G}{\partial z} + G\frac{\partial}{\partial z}\right\}^2 \mathrm{e}^{-L_0 t'}. \tag{A.11}$$

Substituting into (1.22) into (1.17) we obtain

$$\frac{\partial \langle \sigma(t)\rangle_\eta}{\partial t} = D\mathrm{e}^{L_0 t}\left\{\frac{\partial G}{\partial z} + G\frac{\partial}{\partial z}\right\}^2 \mathrm{e}^{-L_0 t}\langle \sigma(t)\rangle_\eta. \tag{A.12}$$

Recall that $\rho(t) = \mathrm{e}^{-L_0 t}\sigma(t)$ and we finally obtain

$$\frac{\partial \langle \rho(t)\rangle_\eta}{\partial t} = -L_0\langle \rho(t)\rangle_\eta + \mathrm{e}^{-L_0 t}\frac{\partial \langle \sigma(t)\rangle_\eta}{\partial t} \tag{A.13}$$

$$= -L_0\langle \rho(t)\rangle_\eta + D\left\{\frac{\partial G}{\partial z} + G\frac{\partial}{\partial z}\right\}^2 \langle \rho(t)\rangle_\eta. \tag{A.14}$$

Hence introducing $P(t) = \langle \rho(t)\rangle_\eta$ we have

$$\frac{\partial P}{\partial t} = D\left\{\frac{\partial G}{\partial z} + G\frac{\partial}{\partial z}\right\}^2 P - L_0 P \tag{A.15}$$

$$= D\left(\frac{\partial}{\partial z}G\frac{\partial}{\partial z}(GP)\right) - \frac{\partial}{\partial z}(FP). \tag{A.16}$$

A steady state solution to this equation can be shown to be

$$P(z) = \frac{1}{Z|G(z)|}\exp[-\Psi(z)] \tag{A.17}$$

$$\Psi(z) = -\frac{1}{D}\int^z \frac{F}{G^2}\mathrm{d}z \tag{A.18}$$

where Z is a normalisation factor.

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